Derivation of Kanade-Lucas-Tomasi Tracking Equation

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January 20, A. D. 1997

Carlo Tomasi [1] has recently proposed the following symmetric definition for the dissimilarity between two windows, one in image I and one in image J:

$$\epsilon = \int\int_W [J(\mathbf{x} + \frac{\mathbf{d}}{2}) - I(\mathbf{x} - \frac{\mathbf{d}}{2})]^2 w(\mathbf{x})\, d\mathbf{x}, \tag{1}$$

where $\mathbf{x} = [x, y]^T$, the displacement $\mathbf{d} = [d_x, d_y]^T$, and the weighting function $w(\mathbf{x})$ is usually set to the constant 1. Equation (1) is identical to the equation given in [2] except that the current version has been made symmetric with respect to both images by replacing $[J(\mathbf{x}) - I(\mathbf{x} - \mathbf{d})]$ with $[J(\mathbf{x} + \frac{\mathbf{d}}{2}) - I(\mathbf{x} - \frac{\mathbf{d}}{2})]$.

Now the Taylor series expansion of J about a point $\mathbf{a} = [a_x, a_y]^T$, truncated to the linear term, is:

$$J(\xi) \approx J(\mathbf{a}) + (\xi_x - a_x)\frac{\partial J}{\partial x}(\mathbf{a}) + (\xi_y - a_y)\frac{\partial J}{\partial y}(\mathbf{a}),$$

where $\xi = [\xi_x, \xi_y]^T$.

Following the derivation in [3], we let $\mathbf{x} + \frac{\mathbf{d}}{2} = \xi$ and $\mathbf{x} = \mathbf{a}$ to get:

$$J(\mathbf{x} + \frac{\mathbf{d}}{2}) \approx J(\mathbf{x}) + \frac{d_x}{2}\frac{\partial J}{\partial x}(\mathbf{x}) + \frac{d_y}{2}\frac{\partial J}{\partial y}(\mathbf{x}).$$

Similarly,

$$I(\mathbf{x} - \frac{\mathbf{d}}{2}) \approx I(\mathbf{x}) - \frac{d_x}{2}\frac{\partial I}{\partial x}(\mathbf{x}) - \frac{d_y}{2}\frac{\partial I}{\partial y}(\mathbf{x}).$$

Therefore,

$$\begin{aligned}
\frac{\partial \epsilon}{\partial \mathbf{d}} &= 2\int\int_W [J(\mathbf{x} + \frac{\mathbf{d}}{2}) - I(\mathbf{x} - \frac{\mathbf{d}}{2})][\frac{\partial J(\mathbf{x} + \frac{\mathbf{d}}{2})}{\partial \mathbf{d}} - \frac{\partial I(\mathbf{x} - \frac{\mathbf{d}}{2})}{\partial \mathbf{d}}] w(\mathbf{x})\, d\mathbf{x}, \\
&\approx \int\int_W [J(\mathbf{x}) - I(\mathbf{x}) + \mathbf{g}^T \mathbf{d}]\mathbf{g}(\mathbf{x}) w(\mathbf{x})\, d\mathbf{x},
\end{aligned}$$

where

$$\mathbf{g} = \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{I+J}{2}\right) & \frac{\partial}{\partial y}\left(\frac{I+J}{2}\right) \end{bmatrix}^T.$$

To find the displacement $\mathbf{d}$, we set the derivative to zero:

$$\frac{\partial \epsilon}{\partial \mathbf{d}} = \int\int_W [J(\mathbf{x}) - I(\mathbf{x}) + \mathbf{g}^T(\mathbf{x})\mathbf{d}]\mathbf{g}(\mathbf{x}) w(\mathbf{x})\, d\mathbf{x} = 0.$$

Rearranging terms, we get

$$\int\int_W [J(\mathbf{x}) - I(\mathbf{x})]\mathbf{g}(\mathbf{x})w(\mathbf{x})\,d\mathbf{x} = -\int\int_W \mathbf{g}^T(\mathbf{x})\mathbf{d}\mathbf{g}(\mathbf{x})w(\mathbf{x})\,d\mathbf{x},$$
$$= -\left[\int\int_W \mathbf{g}(\mathbf{x})\mathbf{g}^T(\mathbf{x})w(\mathbf{x})\,d\mathbf{x}\right]\mathbf{d}.$$

In other words, we must solve the equation

$$Z\mathbf{d} = \mathbf{e}, \tag{2}$$

where Z is the following 2×2 matrix:

$$Z = \int\int_W \mathbf{g}(\mathbf{x})\mathbf{g}^T(\mathbf{x})w(\mathbf{x})\,d\mathbf{x}$$

and e is the following 2×1 vector:

$$\mathbf{e} = \int\int_W [I(\mathbf{x}) - J(\mathbf{x})]\mathbf{g}(\mathbf{x})w(\mathbf{x})\,d\mathbf{x}.$$

References

[1] C. Tomasi. Personal correspondence, May 1996.

[2] C. Tomasi and T. Kanade. Detection and Tracking of Point Features. Carnegie Mellon University Technical Report CMU-CS-91-132, April 1991.

[3] J. Shi and C. Tomasi. Good features to track. In *CVPR*, pages 593-600, 1994.